Lemma 14.29. \( M[G] \) is a ctm of ZFC.

Proof. We show that \( M[G] \) satisfies each axiom of ZFC.

- Extensionality. Follows easily from transitivity of \( M[G] \).

- Regularity. Trivial.

- Pairing. Let \( x, y \in M[G] \); then there exist \( \tau, \sigma \in M^P \) with \( \tau_G = x \) and \( \sigma_G = y \). Now consider the set \( \delta = \{ \langle \tau, 1 \rangle, \langle \sigma, 1 \rangle \} \). It is easy to see that \( \delta_G = \{ \tau_G, \sigma_G \} = \{ x, y \} \). But note that \( \delta \in M^P \): it is a \( P \)-name by construction, and is in \( M \) since \( M \) is a ctm.

- Union. Suppose \( a \in M[G] \). We wish to show that there is some \( b \in M[G] \) which contains \( \bigcup a \) as a subset (we can then appeal to Separation in \( M[G] \), which we will show later).

Since \( a \in M[G] \), there is some \( \tau \in M^P \) with \( \tau_G = a \). Let \( \pi = \bigcup \text{dom}(\tau) \); this is a set which contains the \( P \)-names of all elements of \( \tau_G \) (and possibly some extra ones whose corresponding conditions are not in \( G \) ). \( \pi \in M \) since \( M \) is a ctm; \( \pi \in V^P \) by construction (\( \text{dom}(\tau) \) is a set of \( P \)-names, so \( \bigcup \text{dom}(\tau) \) is a subset of \( V^P \times P \)). Hence \( \pi \in M^P \), so \( \pi_G \in M[G] \).

We claim that \( \bigcup a \subseteq \pi_G \). To see this, let \( c \in a \); then \( c = \sigma_G \) for some \( \sigma \in \text{dom}(\tau) \). Therefore \( \sigma \subseteq \pi \), so \( \sigma_G \subseteq \pi_G \).

- Separation. Let \( \sigma \in M^F \) and let \( \varphi \) be a formula (it may have multiple parameters, but we omit them in the following proof), and define \( c = \{ a \in \sigma_G \mid M[G] \models \varphi[a] \} \).

We wish to show that \( c \in M[G] \), which we will do by finding a suitable \( P \)-name for \( c \).

We claim that a suitable \( P \)-name is \( \rho = \{ \langle \pi, p \rangle \in \text{dom}(\sigma) \times P \mid p \models \varphi[\pi_G] \} \).

We first note that \( \rho \in M \) by separation in \( M \) and definability of \( \models \) (Theorem 14.27); \( \rho \) is clearly a \( P \)-name by construction. Now we must show that \( \rho_G = c \).

- (\( \rho_G \subseteq c \)). Suppose \( x \in \rho_G \), so there is some \( \langle \pi, p \rangle \in \rho \) such that \( x = \pi_G \) and \( p \models \varphi[\pi_G] \). Then by definition of forcing, \( \pi_G \in \sigma_G \) and \( M[G] \models \varphi[\pi_G] \). Hence \( x = \pi_G \in c \) by definition of \( c \).
\( c \subseteq \rho_G \). Suppose \( a \in c \), that is, \( a \in \sigma_G \) and \( M[G] \models \varphi[a] \). Then there is some \( \pi \in M^P \) such that \( \pi_G = a \). So by Truth (Theorem 14.26) we may pick \( p \in G \) such that \( p \models \pi \wedge \varphi(\pi) \). Then \( (\pi, p) \in \rho \), so \( a = \pi_G \in \rho_G \).

- Replacement. At this point, we introduce the axiom schema of Collection:

\[
\forall x. \exists y. \forall z \in x. (\exists w. \varphi(z, w) \Rightarrow \exists w \in y. \varphi(z, w)).
\]

Intuitively, this states that we can collect elements in the image of any set \( x \) under any partial relation \( \varphi \) into a set \( y \) (which may also contain other stuff). This implies the axiom schema of Replacement: we may take \( \varphi \) to be a functional relation, and then given a set \( y \) witnessing Collection, we may use Separation to yield a set which is exactly the image \( \varphi[x] \).

It turns out that Collection is also a theorem of ZF, via the reflection principle.

Now suppose we have some \( x = \sigma_G \); we wish to exhibit a \( \rho \) for which

\[
M[G] \models \forall z \in \sigma_G. (\exists w. \varphi(z, w) \Rightarrow \exists w \in \rho_G. \varphi(z, w)).
\]  

(2)

Let \( S \in M \) such that

\[
M \models \forall \pi \in \text{dom}(\sigma). \forall p \in P. (\exists \mu. M^P(\mu) \wedge p \models \varphi(\pi, \mu) \Rightarrow (\exists \mu \in S). p \models \varphi(\pi, \mu)).
\]

(3)

It is not \textit{a priori} clear that such an \( S \) exists. If \( M^P \) were a set, we could just take \( S = M^P \), but \( M^P \) may be a proper class. However, such an \( S \) does exist, which we can show as follows (note that in the following, all our reasoning is taking place inside \( M \)). By Reflection in \( M \), there is a closed unbounded class of ordinals \( \alpha \) which simultaneously reflect the two formulae

\[
\exists \mu. M^P(\mu) \wedge p \models \varphi(\pi, \mu)
\]

and

\[
M^P(\mu) \wedge p \models \varphi(\pi, \mu),
\]

that is,

\[
\forall \pi \in \text{dom}(\sigma). \forall p \in P. (\exists \mu. M^P(\mu) \wedge p \models \varphi(\pi, \mu) \Leftrightarrow [\exists \mu. M^P(\mu) \wedge p \models \varphi(\pi, \mu)]^{V_\alpha}),
\]  

(3)

and

\[
\forall \pi \in \text{dom}(\sigma). \forall p \in P. \forall \mu. (M^P(\mu) \wedge p \models \varphi(\pi, \mu) \Leftrightarrow [M^P(\mu) \wedge p \models \varphi(\pi, \mu)]^{V_\alpha}).
\]  

(4)

So, we may pick such an \( \alpha \) large enough so that \( \text{dom}(\sigma) \in V_\alpha \) and \( P \in V_\alpha \).
We then let $S = MP \cap V_\alpha$, and claim that $S$ has the required property.
Given some $\pi \in \text{dom}(\sigma)$ and $p \in P$, suppose there exists some $\mu \in MP$ for which $p \vDash \varphi(\pi, \mu)$. Then by equation (3) there is some $\mu \in V_\alpha$ which satisfies $[MP(\mu) \land p \vDash \varphi(\pi, \mu)]^{V_\alpha}$; but then by equation (4) $\mu$ also satisfies this condition in the universe, so $\mu \in S$ and $p \vDash \varphi(\pi, \mu)$, exactly the required property of $S$.

Now let $\rho = S \times \{1_P\}$, so $\rho_G = \{\mu_G | \mu \in S\}$ (since $G$ is a filter). Now we must show that $\rho$ satisfies equation (2).

To this end, let $z \in \sigma_G$ and $\varphi^M[G](z, w)$ for some $w \in M[G]$. We must find some $w' \in \rho_G$ for which $\varphi^M[G](z, w')$.

Since $z \in \sigma_G$, $z = \pi_G$ for some $\pi \in \text{dom}(\sigma)$. We know that $M[G] \models \exists w. \varphi(\pi_G, w)$, so there must be some $\mu$ for which $M[G] \models \varphi(\pi_G, \mu_G)$. Then by Truth there is some $p \in G$ such that $p \vDash \varphi(\pi, \mu)$. Then by the property of $S$, there is some $\mu' \in S$ such that $p \vDash \varphi(\pi, \mu')$, and $\mu'_G \in \rho_G$.

Remark. We are not quite done; in the next lecture we will cover Powerset and Choice. But now, a small digression about the axiom schema of Collection.

Definition 14.30. Kripke-Platek set theory is the axiomatic system with Extensionality, Regularity, Pairing, Union, and all $\Delta_0$ instances of Separation and Collection.

Remark. It is easy to see that $V_\omega \models KP$, since it models $ZF - \text{Infinity}$. $KP + \text{Infinity}$ is a nice system, too.

Definition 14.31. An ordinal $\alpha$ is admissible iff $L_\alpha \models KP$.

Remark. Admissible ordinals “are those which support a nice notion of computability.”

Definition 14.32. $R \subseteq \omega \times \omega$ is recursive iff $c_R$, the characteristic function of $R$, is Turing-computable. An ordinal $\alpha$ is recursive iff it is the order type of some recursive $R \subseteq \omega \times \omega$.

Definition 14.33. $\omega^1_{CK}$, the Church-Kleene ordinal, is the least non-recursive ordinal.

$(\omega^1_{CK})^f$ is the least non-(recursive)$^f$ ordinal, where $f \in \omega \rightarrow 2$ and (recursive)$^f$ means Turing-computable given an $f$-oracle.

Theorem 14.34. If $\alpha$ is a countable ordinal greater than $\omega$, then $\alpha$ is admissible iff $\alpha = (\omega^1_{CK})^f$ for some $f \in \omega \rightarrow 2$.

Remark. The proof is omitted.

We note that $\omega^1_{CK}$ is, in fact, the set of all recursive ordinals, so in particular it must be countable (since there are countably many Turing machines).