Lemma 14.14. If $Y$ is countable, then $FP(X, Y)$ has the ccc.

Proof. Suppose $Y$ is countable and consider any uncountable set of finite partial functions

$$P = \{ p_{\alpha} \mid \alpha < \aleph_1 \} \subseteq FP(X, Y).$$

We wish to show that $P$ is not an antichain.

Let $Z = \text{dom}[P]$. By Lemma 14.13, there is some $Z' \subseteq Z$ which is uncountable and quasi-disjoint. Let $d$ be the common intersection of the elements of $Z'$, and consider the set of functions $\overrightarrow{d}Y$. This set is countable since $Y$ is countable and $d$ is finite.

For $p, q \in FP(X, Y)$, define $p \sim q$ iff $p \upharpoonright d = q \upharpoonright d$, and $P' = \{ p_{\alpha} \mid \text{dom}(p_{\alpha}) \in Z' \}$. Consider $P'/\sim$: each equivalence class is represented by some function $d \to Y$, so there are countably many equivalence classes. However, $P'$ is uncountable, so there must be some uncountable equivalence class, call it $B$. But any two $p, q \in B$ are compatible, since they agree on $d$, the intersection of their domains. Hence $P$ is not an antichain: in fact, it must contain uncountably many compatible elements!

Lemma 14.15 (Approximation Lemma). If $(P \text{ has the ccc})^M$, $M$ is a ctm, $X, Y \in M$ and $f : X \to Y \in M[G]$, then there is an $F : X \to P(Y) \in M$ such that for every $a \in X$, $f(a) \in F(a)$ and $(F(a) \text{ is countable})^M$.

Remark. This lemma essentially says that given any function $f \in M[G]$, we may “approximate” it in $M$, even though $f$ itself may not be an element of $M$. We defer the proof of this lemma to the remainder of the semester.

Lemma 14.16. If $(P \text{ has the ccc})^M$ and $M$ is a ctm, then $\text{Card}^M(\kappa)$ implies $\text{Card}^M[G](\kappa)$.

Remark. Note that $\text{Card}(\kappa)$ denotes “$\kappa$ is a cardinal”; not to be confused with $\text{card}(\kappa)$, the cardinality of $\kappa$. We also note that this lemma is only interesting for uncountable $\kappa$, since finite cardinals and $\omega$ are absolute; we don’t have to worry about those getting collapsed in $M[G]$.

Proof. Suppose, by way of contradiction, that $\text{Card}^M(\kappa)$ but there is some infinite $\beta < \kappa$ and some $f \in M[G]$ with $f : \beta \to \kappa$.

By Lemma 14.15, there is some $F : \beta \to P(\kappa) \subseteq M$ for which $\bigcup \text{rng}(F) = \kappa$. But now ($\text{card}(\kappa) = \kappa = \text{card}(\bigcup \text{rng}(F)) \leq \text{card}(\beta) \times \aleph_0 = \text{card}(\beta) < \kappa)^M$, a contradiction.

Definition 14.17. $\tau$ is a $P$-name iff $\tau$ is a relation and for every $\langle \sigma, p \rangle \in \tau$, $\sigma$ is a $P$-name and $p \in P$. 65
Remark. This definition might seem circular, but we can formalize it by induction on the transitive closure of $\tau$.

**Definition 14.18.** Suppose $\tau$ is a $P$-name and $G \subseteq P$. Then define

$$\text{val}(\tau, G) = \{ \text{val}(\sigma, G) \mid \exists p \in G. (\sigma, p) \in \tau \}.$$ 

**Definition 14.19.** $V^P$ denotes the class of all $P$-names. $M^P$ denotes $M \cap V^P$, which is equal to $(V^P)^M$ because of some lemma about recursion and absoluteness.

**Remark.** Let’s look quickly at a few examples.

- Of course, $\emptyset \in V^P$ trivially; $\text{val}(\emptyset, G) = \emptyset$ for all $G$.
- Also, consider $\tau = \{ (\emptyset, p) \} \in V^P$. We have

$$\text{val}(\tau, G) = \begin{cases} \{ \emptyset \} & \text{if } p \in G \\ \emptyset & \text{otherwise.} \end{cases}$$

- $\rho = \{ (\emptyset, 1_P) \}$ is also a valid $P$-name; $\text{val}(\rho, G) = \{ \emptyset \}$ for all filters $G$.
- We may generalize this to

$$\dot{x} = \{ (\dot{y}, 1_P) \mid y \in x \}.$$ 

We can consider $\dot{x}$ to be a “canonical name” for $x$: $\text{val}(\dot{x}, G) = x$ for every filter $G$.

**Definition 14.20.** Given a ctm $M$, $P \in M$, and a $G$ which is $P$-generic over $M$, define

$$M[G] = \{ \text{val}(\tau, G) \mid \tau \in M^P \}.$$ 

**Remark.** By the above remark concerning canonical names, we observe that $M \subseteq M[G]$. 

66