Remark. We now proceed to prove the generalized continuum hypothesis under the assumption that $V = L$.

**Lemma 13.24.** For every infinite ordinal $\alpha$, $\text{card}(L_\alpha) = \text{card}(\alpha)$.

**Proof.** First, we note that $L_\omega = V_\omega$, and $\text{card}(V_\omega) = \text{card}(\omega) = \omega$. Also, it is clear that $\text{card}(L_\alpha) \geq \text{card}(\alpha)$ since $\alpha \subseteq L_\alpha$.

In the successor case, we want to show that $\text{card}(L_{\alpha+1}) = \text{card}(\text{Def}(L_\alpha)) = \text{card}(\alpha + 1) = \text{card}(\alpha)$. This amounts to showing that $\text{Def}$ preserves cardinality. But every element of $\text{Def}(L_\alpha)$ is a formula together with some finite number of witnesses from $L_\alpha$; hence its size is at most $\aleph_0 \times \sum_{n \in \omega} (\text{card}(L_\alpha))^n = \text{card}(L_\alpha)$. \qed

**Definition 13.25.** $o(M)$ is the least $\gamma$ for which $\gamma \notin M$. For transitive $M$, $o(M) = \{ \gamma \mid \gamma \in M \}$.

Remark. Recall that the GCH says that $2^\kappa = \kappa^+$ for all infinite $\kappa$. To show that it holds in $L$, we must show that every subset of $L_\kappa$ occurs at some level prior to $L_{\kappa^+}$. If we can show that $\text{od}(x) < \kappa^+$ for every $x \leq \kappa$, then $2^\kappa \leq \kappa^+$ (we already know that $2^\kappa \geq \kappa^+$ by Cantor’s Theorem). In particular, we will show that for every $x \subseteq L_\alpha$, $\text{od}(x) < |\alpha|^+$.

Recall that $\alpha \mapsto L_\alpha$ is $\Delta_1$-ZF. So there is some sentence $\theta$ for which $\alpha \mapsto L_\alpha$ is $\Delta_1$-$\theta$, that is, $\theta$ proves the equivalence of the $\Sigma_1$ and $\Pi_1$ forms of $\alpha \mapsto L_\alpha$. Given this, we can prove the following lemma.

**Lemma 13.26.** There is a sentence $\theta$ such that $\text{ZF} + (V = L) \vdash \theta$ and for every transitive $M$,

$$M \models \theta \implies \exists \alpha. \text{lim}(\alpha) \land M = L_\alpha.$$ 

**Proof.** Let $\psi$ be the function $\alpha \mapsto L_\alpha$. Then $\psi(\alpha, x)$ is absolute for transitive models of a finite fragment $\theta'$ of ZF. Then let

$$\theta = \theta' \land (V = L).$$

If $M \models \theta$, the claim is that $M = L_\alpha$ for some $\text{lim}(\alpha)$.

In particular, we claim that $M = L_{o(M)}$.

- Since $\alpha \mapsto \alpha + 1$ is absolute for $M$, $o(M)$ must be a limit.

- $L_\alpha \subseteq M$. Since $\text{lim}(\alpha)$, $L_\alpha = \bigcup_{\beta < \alpha} L_\beta$, and $\beta \in M$ for all $\beta < \alpha$. $\psi(\beta, x)$ is absolute for $M$, so $L_\beta \subseteq M$ for all $\beta < \alpha$. Hence, $\bigcup_{\beta < \alpha} L_\beta \subseteq M$ by transitivity of $M$.
\[ M \subseteq L_\alpha. \] Note that \( M \models V = L \). For \( \beta < \alpha \), \( (L_\beta)^M = L_\beta \); hence \( M \subseteq \bigcup_{\beta < \alpha} L_\beta \).

\[ \square \]

**Theorem 13.27.** For every \( x, \alpha \), if \( L(x) \) and \( x \subseteq L_\alpha \) then there is some \( \beta < |\alpha|^+ \) with \( x \in L_\beta \).

**Remark.** We first remark that this theorem implies the GCH; note that if \( x \subseteq \kappa \) then \( x \subseteq L_\kappa \). This theorem says that every subset of \( L_\alpha \) gets constructed at some stage prior to \( |\alpha|^+ \); hence the set of all such subsets must occur at stage \( |\alpha|^+ \).

**Proof.** Observe that \( \theta \) is a consequence of \( V = L \). Since \( \theta \) is a single sentence, we can apply the Reflection Principle.

Suppose \( x \subseteq L_\alpha \) and \( L(x) \); hence there is some \( \delta \) with \( x \in L_\delta \). Pick \( \beta > \delta \), \( \beta > \alpha \), \( \lim(\beta) \) from the club class of ordinals reflecting \( \theta \) in \( L \). Hence \( x \in L_\beta \), and we note that \( L_\alpha \subseteq L_\beta \).

Since AC holds in \( L \), by the Löwenheim-Skolem theorem there is some \( N \preceq L_\beta \) such that \( L_\alpha \cup \{x\} \subseteq N \) and \( |N| = \text{card}(\alpha) \); we also note that \( N \models \theta \) since \( N \preceq L_\beta \). Also, observe that \( L_\alpha \cup \{x\} \) is transitive, since \( x \subseteq L_\alpha \). (However, \( N \) might not be transitive.)

But \( N \) is extensional and well-founded, so by the Mostowski collapsing theorem, it is isomorphic to a unique transitive set \( M \), and the isomorphism preserves \( L_\alpha \cup \{x\} \) (the Mostowski isomorphism is the identity on any transitive sets).

Hence \( M \models \theta \) since it is isomorphic to \( N \). So \( M = L_\gamma \), \( \lim(\gamma) \), with \( \alpha < \gamma < |\alpha|^+ \) (\( \alpha < \gamma \) since \( L_\alpha \subseteq L_\gamma \); \( \gamma < |\alpha|^+ \) since \( M \) has cardinality \( \alpha \)).

\[ \square \]