Lemma 5.20. Suppose that $X$ is a collection of sets, and that $|X| = \kappa$ and $\sup\{|Z| \mid Z \in X\} = \lambda$. Then $|\bigcup X| \leq \kappa \times \lambda$.

Proof. By the well-ordering principle (AC), we can make an enumeration of $X$,

$$X = \{ Z_\alpha \mid \alpha < \kappa \}.$$ 

For each $\alpha$, $|Z_\alpha| = \lambda_\alpha < \lambda$. Again by the well-ordering principle, we can make an enumeration of each $Z_\alpha$,

$$Z_\alpha = \{ u_{\alpha\beta} \mid \beta < \lambda_\alpha \}.$$ 

Then we can write $\bigcup X$ as

$$\bigcup X = \{ u_{\alpha\beta} \mid \alpha < \kappa, \beta < \lambda_\alpha \},$$

which clearly has cardinality at most $\kappa \times \lambda$.

Remark. This result is in some sense a generalization of the fact that $\mathbb{Q}$ is countable, with one important difference. To show that the rationals are countable, we just have to exhibit a bijection between the rationals (or, more simply, between $\mathbb{N} \times \mathbb{N}$ and the naturals. From this result, it seems like it should follow that if $X$ is a countable collection of countable sets, then $\bigcup X$ is also countable; but to show this, we need the AC (which we don’t need to show the countability of $\mathbb{Q}$). Intuitively, this is because we need to be able to “pick” an ordering for each $Z \in X$.

The above result is more general yet: instead of talking about a countable union of countable sets, is about a cardinality-$\kappa$ union of sets with cardinality at most $\lambda$; the fact about countable sets in particular follows from the fact that $\omega \times \omega = \omega$.

Lemma 5.21. For every ordinal $\alpha$, there exists a strictly increasing cofinal map from $\text{cf}(\alpha)$ to $\alpha$.

Proof. Let $g : \text{cf}(\alpha) \to \alpha$ be a cofinal map. Then define $f : \text{cf}(\alpha) \to \alpha$ by

$$f(\beta) = \max\{g(\beta), \sup_{\gamma < \beta} (f(\gamma) + 1)\}.$$ 

By definition, $\sup(\text{rng}(f)) \geq \sup(\text{rng}(g)) = \alpha$, so $f$ is cofinal. $f$ is also strictly increasing: if $\beta > \gamma$, then $f(\beta) > \sup_{\gamma < \beta} f(\gamma) \geq f(\gamma)$. 

Lemma 5.22. $\text{cf}$ is idempotent.
Proof. Let $\alpha$, $\beta$, and $\gamma$ be ordinals such that $\text{cf}(\alpha) = \beta$ and $\text{cf}(\beta) = \gamma$. By Lemma 5.21, suppose $f : \beta \to \alpha$ and $g : \gamma \to \beta$ are strictly increasing cofinal maps. Let $\delta \in \alpha$. Since $f$ is a cofinal map into $\alpha$, there must be some $\zeta \in \beta$ for which $f(\zeta) > \delta$. Likewise, there must be some $\eta \in \gamma$ for which $g(\eta) > \zeta$. Since $f$ is strictly increasing, we conclude that $f(g(\eta)) > f(\zeta) > \delta$; hence $f \circ g$ is a cofinal map into $\alpha$, and $\beta = \gamma$.

Lemma 5.23. If $\alpha > 0$ is a limit ordinal, then $\text{cf}(\alpha)$ is an infinite, regular cardinal.

Proof. By definition, $\text{cf}(\alpha)$ is the least $\beta$ for which there exists a cofinal map $f : \beta \to \alpha$ (that is, for which $\sup(\text{rng}(f)) = \alpha$). Suppose that $\text{cf}(\alpha)$ is not a cardinal. Then there exists some $\gamma < \text{cf}(\alpha)$ such that $\gamma \sim \text{cf}(\alpha)$, that is, there exists some $g : \gamma \overset{1-1}{\longrightarrow} \text{cf}(\alpha)$. But then $f \circ g : \gamma \to \alpha$ is also a cofinal map, contradicting the minimality of $\text{cf}(\alpha)$. Also, $\text{cf}(\alpha)$ must be infinite since there cannot exist a cofinal map from a finite set into an infinite one; $\text{cf}(\alpha)$ is regular by Lemma 5.22.

Theorem 5.24. For every $\kappa \geq \omega$, $\kappa^+$ is regular. That is, $\aleph_{\alpha+1}$ is regular for all $\alpha$.

Remark. To help provide some intuition for the relationship of this theorem to Lemma 5.20, we can show the following special case, namely, that $\omega^+ = \aleph_1$ is regular.

Suppose otherwise, namely, that $\text{cf}(\aleph_1) = \omega$ (note, by Lemma 5.23, that this is the only choice for $\text{cf}(\aleph_1)$ if $\aleph_1$ is not regular). That is, for some $f : \omega \to \aleph_1$, $\text{rng}(f)$ is cofinal in $\aleph_1$, i.e., $\bigcup \text{rng}(f) = \aleph_1$. Now, we note the following facts:

- $|\text{rng}(f)| = \omega$. This is clear since $\text{dom}(f) = \omega$.
- For every $\alpha \in \text{rng}(f)$, $|\alpha| \leq \omega$. This follows since $\alpha \in \aleph_1$, so the biggest its cardinality could possibly be is $\aleph_0 = \omega$.

Hence $\bigcup \text{rng}(f)$ is a countable union of countable sets—but we know this is countable, so it cannot be equal to $\aleph_1$.

Proof. We now give a general proof of Theorem 5.24; it follows much the same shape as the preceding remark.

For purposes of contradiction, suppose that $\aleph_{\alpha+1}$ is not regular, that is, there is some cofinal map $f : \aleph_\beta \to \aleph_{\alpha+1}$ where $\beta \leq \alpha$. Then $\bigcup \text{rng}(f) = \aleph_{\alpha+1}$. If $\gamma \in \text{rng}(f)$, then $|\gamma| < \aleph_{\alpha+1}$. Therefore, $|\text{rng}(f)| = \aleph_\beta$ and $\sup(\text{rng}(f)) = \aleph_\alpha$, so by Lemma 5.20, the cardinality of $\bigcup \text{rng}(f)$ is $\alpha \times \beta = \max(\alpha, \beta) < \alpha + 1$, contradicting the cofinality of $f$.

Remark. Theorem 5.24 asserts that all successor cardinals are regular. However, it turns out that we can’t even prove that there exist any regular limit cardinals (i.e., weakly inaccessible cardinals) other than $\omega$!
Recall that the Continuum Hypothesis posits that there is no cardinal intermediate between \( \omega \) and \( |\mathcal{P}(\omega)| \); that is, there does not exist a set \( X \) such that \( \omega < |X| < |\mathcal{P}(\omega)| \). Given the AC, we can reformulate this as the equality

\[
2^{\aleph_0} = \aleph_1,
\]

that is, \( |\mathcal{P}(\omega)| \) is the next cardinal after \( \aleph_0 \).

This suggests what is known as the Generalized Continuum Hypothesis (GCH):

\[
\forall \kappa, 2^\kappa = \kappa^+.
\]

Note that in the presence of the GCH, “weakly inaccessible” and “strongly inaccessible” are equivalent.

Using the model of constructible sets, Gödel in 1939 showed that ZF + AC + GCH is consistent if ZF is; it’s relatively clear what this system would look like. However, Cohen showed that ZFC + \( \neg \)CH is consistent if ZF is; what does ZFC look like with \( \neg \)CH? In fact, it turns out that for every \( \alpha \geq 0 \), ZFC + (\( 2^{\aleph_0} = \aleph_{\alpha+1} \)) is consistent if ZF is! That is, \( 2^{\aleph_0} \) could be \( \aleph_1 \), or \( \aleph_2 \), or \( \aleph_\omega \), but it could not be \( \aleph_\omega \) or \( \aleph_{\omega+\omega} \), and so on.

Let’s prove that \( \text{cf}(2^{\aleph_0}) > \omega \). Strangely enough, in light of the previous remarks, this is just about all we can say about \( 2^{\aleph_0} \)! This will follow as a corollary to Theorem 5.27.

**Definition 5.25.** Given a collection of sets \( X_i \) indexed by the elements of some set \( I \), we may form the sum

\[
\sum_{i \in I} X_i = \bigcup_{i \in I} (X_i \times \{i\}),
\]

that is, the disjoint union of all the \( X_i \)'s, using the indices as tags.

**Definition 5.26.** Given a collection of sets \( X_i \), we may also form the product

\[
\prod_{i \in I} X_i = \{ f : I \rightarrow \bigcup_{i \in I} X_i \mid \forall i \in I, f(i) \in X_i \}.
\]

That is, \( \prod_{i \in I} X_i \) is the set of functions which pick out an element of \( X_i \) for each \( i \in I \). As an example, \( \mathbb{R}^3 = \prod_{i \in \{0, 1, 2\}} \mathbb{R} \) is the set of functions that pick out a real number for each of the three indices 0, 1, and 2; these can also be thought of as ordered triples (although they are not actually triples in a technical sense).

**Theorem 5.27** (König, 1905). Suppose that \( \forall i \in I, \kappa_i < \lambda_i \). Then

\[
\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i.
\]

**Remark.** We defer the proof of Theorem 5.27 to examine two corollaries.

**Corollary 5.28** (Cantor’s Theorem (Theorem 5.5)).
Proof. Let $\kappa_i = 1$ and $\lambda_i = 2$. Then $\sum_{i \in I} 1 \sim I$, and $\prod_{i \in I} 2 \sim \mathcal{P}(I)$. \hfill \Box

Corollary 5.29. $\text{cf}(2^{\aleph_0}) > \omega$.

Proof. Let $f : \omega \to 2^{\aleph_0}$, and for $i \in \omega$ let $\kappa_i = |f(i)|$; thus $\kappa_i < 2^{\aleph_0}$. Also, let $\lambda_i = 2^{\aleph_0}$, for all $i$. Then

$$
\sup_{i \in \omega} \kappa_i \leq \sum_{i \in \omega} \kappa_i < \prod_{i \in \omega} \lambda_i = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0}.
$$

\hfill \Box

Proof. We now prove Theorem 5.27. Suppose we have a family of sets $Z_i$, and let $\lambda_i = |Z_i|$ and $\kappa_i < \lambda_i$ for $i \in I$. Now let $Z = \prod_{i \in I} Z_i$, and for each $i \in I$ pick (by the AC) some $Y_i \subset Z$ with $|Y_i| = \kappa_i$. Then we will show that $\bigcup_{i \in I} Y_i \neq Z$, from which the theorem follows immediately.

For each $i \in I$, define $w_i = \{ g(i) \mid g \in Y_i \}$. Clearly $|w_i| \leq \kappa_i < \lambda_i$. Therefore, $V_i = Z_i - w_i \neq \emptyset$, and $\prod_{i \in I} V_i \neq \emptyset$. (We note in passing that this is another formulation of the AC—that the product of a nonempty collection of nonempty sets is nonempty.)

But $\prod_{i \in I} V_i \subseteq Z$ is disjoint from $\bigcup_{i \in I} Y_i$; hence $\bigcup_{i \in I} Y_i \neq Z$. \hfill \Box

Remark. This is a generalized “diagonal” argument, which explains why Cantor’s Theorem follows so readily as a corollary. Some additional commentary should go here.