4 Ordinals

The ordinals are canonical well-ordered sets.

**Definition 4.1.** A set $x$ is transitive iff $\forall y \forall z. y \in x \Rightarrow y \subseteq x$.

*Remark.* If $z$ is transitive, then $x \in y \in z \Rightarrow x \in z$.

**Definition 4.2.** $x$ is an ordinal iff

- $x$ is transitive, and
- $\langle x, \in \rangle \restriction x$ is a well-ordering.

*Remark.* In what follows, we use $\alpha$, $\beta$, and $\gamma$ to refer to arbitrary ordinals.

**Lemma 4.3.** If $x \in \alpha$, then $x$ is an ordinal.

*Proof.* Since $\alpha$ is transitive, $x \subseteq \alpha$; therefore it is clear that $\langle x, \in \rangle \restriction x$ is a well-ordering since $\langle \alpha, \in \rangle \restriction \alpha$ is. To see that $x$ is transitive, suppose the contrary. That is, suppose there is some $y \in x$ and $z \in y$ such that $z \not\in x$. Note that $x$, $y$, and $z$ are all elements of $\alpha$, since $\alpha$ is transitive. Since $\alpha$ is well-ordered under $\in$, either $x = z$ or $x \in z$. If $x = z$, then $z \in y \in z$, contradicting the fact that $\alpha$ is well-ordered; if $x \in z$, then $x \in z \in y \in x$, contradicting the fact that $x$ is well-ordered.

**Lemma 4.4.** If $\beta \subseteq \alpha$ and $\beta \neq \alpha$ then $\beta \in \alpha$.

*Proof.* Consider the set $\gamma = \alpha \cap \beta$, which is nonempty by the given premises. Let $\gamma$ be the $\in$-least element of $\alpha - \beta$. Then $\beta = \gamma$, which can be shown as follows.

1. ($\subseteq$). Suppose there is some element $x \in \beta$ for which $x \not\in \gamma$. Since $x$ and $\gamma$ are both elements of $\alpha$, we must therefore have $\gamma \leq x \in \beta$. Since $\beta$ is transitive, this implies that $\gamma \in \beta$, a contradiction.

2. ($\supseteq$). Suppose $x \in \gamma$; then we must also have $x \in \beta$, since otherwise it would be an element of $\alpha - \beta$ less than $\gamma$, contradicting the definition of $\gamma$.

**Lemma 4.5.** For every $\alpha$, $\beta$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

*Proof.* Suppose otherwise. Consider $\gamma = \alpha \cap \beta$, which by assumption is a proper subset of both $\alpha$ and $\beta$. It is easy to check that $\gamma$ is an ordinal. But then by Lemma 4.4, $\gamma \in \alpha$ and $\gamma \in \beta$, so $\gamma \in \alpha \cap \beta = \gamma$, a contradiction.

**Theorem 4.6.** The class of ordinals is well-ordered by $\in$.

*Proof.* This follows directly from Lemmas 4.4 and 4.5.
Theorem 4.7. For every set $x$ there is an $\alpha$ such that $\alpha \notin x$.

Proof. The proof of this theorem is the Burari-Forti paradox. Suppose there is a set $x$ of which every ordinal is an element. Then by comprehension we may form the set

$$\text{ord} = \{ \alpha \in x \mid \alpha \text{ is an ordinal} \}.$$ 

But by Theorem 4.6 we can see that ord is well-ordered; by Lemma 4.3 it is transitive; hence, $\text{ord} \in \text{ord}$, a contradiction.

Remark. Theorem 4.7 can equivalently be stated as “the class of ordinals is a proper class.”

Some examples of ordinals:

$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

can all easily be checked to be ordinals. Also, if $\alpha$ is an ordinal, then $\alpha \cup \{\alpha\}$ is also.

Definition 4.8. The successor of $\alpha$, denoted $\alpha + 1$, is $\alpha \cup \{\alpha\}$.

Theorem 4.9. $\alpha + 1$ is an ordinal. Moreover, it is the least ordinal bigger than $\alpha$.

Proof. It is easy to see that $(\alpha \cup \{\alpha\}, \in)$ is a strict linear order: for any $x, y \in \alpha \cup \{\alpha\}$, with $x \neq y$, either $x, y \in \alpha$ (in which case $x \in y$ or $y \in x$), or one of $x, y$ is equal to $\alpha$ and the other is an element of $\alpha$. That every non-empty subset has an $\in$-least member follows easily. To see that $\alpha \cup \{\alpha\}$ is transitive, it suffices to note that $\alpha \subseteq \alpha \cup \{\alpha\}$.

To show that $\alpha + 1$ is the least ordinal bigger than $\alpha$, suppose that $\beta > \alpha$. Then by definition, $\alpha \in \beta$, and therefore $\alpha \subseteq \beta$; so $\alpha + 1 = \alpha \cup \{\alpha\} \subseteq \beta$. By Lemma 4.4, $\alpha + 1 \leq \beta$.

Definition 4.10. $\alpha$ is a successor ordinal iff $\alpha = \beta + 1$ for some $\beta$. Otherwise, $\alpha$ is a limit ordinal.

Definition 4.11. The smallest non-zero limit ordinal is called $\omega$ (and it exists by the Axiom of Infinity). The elements of $\omega$ are called natural numbers.

Definition 4.12. $x \sim y$ iff there exists a functional relation which is a 1-1, onto mapping from $x$ to $y$.

Definition 4.13. A set $x$ is finite iff there exists some $n \in \omega$ for which $x \sim n$.

Theorem 4.14. For every well-ordering $(x, <)$ there is an ordinal $\alpha$ such that $(x, <)$ is isomorphic to $(\alpha, \in \mid \alpha)$.

Proof. XXX finish me! 

Theorem 4.15 (Transfinite Induction). If
1. $\varphi(\emptyset)$,

2. $\varphi(\alpha) \implies \varphi(\alpha + 1)$, and

3. $\lim(\lambda) \land (\forall \beta. \beta < \lambda \implies \varphi(\beta)) \implies \varphi(\lambda)$,

then $\forall \beta. \varphi(\beta)$.

Proof. Suppose not; let $\gamma$ be the $\in$-minimal ordinal for which $\neg \varphi(\gamma)$. A simple argument by cases (whether $\gamma$ is $\emptyset$, a successor ordinal, or a limit ordinal) shows that $\gamma$ cannot exist.  \qedsymbol